Examples and Proofs for the paper Functions as processes: termination and the $\bar{\lambda}\mu\tilde{\mu}$-calculus

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1 Example of a term encoded by $[\![\cdot]\!]$

In this section we provide an example of encoded term and we show its reduction steps. Let us consider the following command

$$c = \mu\beta.{<}\,\lambda x.\lambda y.\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\mid \lambda x.\,x\cdot\beta\,{>}\mid M\cdot\alpha\,{>}$$

The command c is the encoding of the λ-term $\lambda x.\lambda y.(x\ y)\ \lambda x.\,x\ M'$ given by the call-by-name compilation of [1]. We consider M as the encoding of M'. The example is complex enough to show clearly how the encoding simulates $\bar{\lambda}\mu\tilde{\mu}$. In what follows we write $env(x,P)$, with x a variable and P a process, the π-term $!x.P$. For sake of readability we omit the initial restrictions (ν-operations) over the channels: every channel is considered private with the exception of μ, $\tilde{\mu}$ and λ.

$[\![c]\!] = Rec\ Y.\ \overline{\mu}(\beta).[\![{<}\,\lambda x.\lambda y.\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\mid \lambda x.\,x\cdot\beta\,{>}]\!] + \tilde{\mu}(x).!x.Y \ \mid\ Rec\ Y.\ \overline{\lambda}(\delta).([\![M]\!] \mid !\delta.[\![\alpha]\!]) + \mu(\beta).!\beta.Y \rightarrow^{\pi}\ (by\ \mu)$

$Rec\ Y.\ \lambda(\delta).[\![\tilde{\mu}x.{<}\,\lambda y.\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\ \mid\ \delta\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\lambda}(\delta).([\![\lambda x.x]\!] \mid !\delta.[\![\beta]\!]) + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \rightarrow^{\pi}\ (by\ \lambda)$

$Rec\ Y.\ \lambda(\delta).[\![\tilde{\mu}x.{<}\,x \mid \delta\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\tilde{\mu}}(x).[\![{<}\,\lambda y.\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\ \mid\ \delta\,{>}]\!] + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \rightarrow^{\pi}\ (by\ \tilde{\mu})$

$Rec\ Y.\ \lambda(\delta').[\![\tilde{\mu}y.{<}\,\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\mid \delta'\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\delta} + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \rightarrow^{\pi}\ (by\ \delta)$

$Rec\ Y.\ \lambda(\delta').[\![\tilde{\mu}y.{<}\,\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\mid \delta'\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\beta} + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \rightarrow^{\pi}\ (by\ \beta)$

$Rec\ Y.\ \lambda(\delta').[\![\tilde{\mu}y.{<}\,\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\mid \delta'\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\lambda}(\delta).([\![M]\!] \mid !\delta.[\![\alpha]\!]) + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \rightarrow^{\pi}\ (by\ \lambda)$

$[\![M]\!] \ \mid\ Rec\ Y.\ \overline{\tilde{\mu}}(y).[\![{<}\,\mu\gamma.{<}\,x \mid y\cdot\gamma\,{>}\mid \delta'\,]\!] + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \rightarrow^{\pi}\ (by\ \tilde{\mu})$

$Rec\ Y.\ \overline{\mu}(\gamma).[\![{<}\,x \mid y\cdot\gamma\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\delta'} + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \rightarrow^{\pi}\ (by\ \delta') \qquad (*)$

$Rec\ Y.\ \overline{\mu}(\gamma).[\![{<}\,x \mid y\cdot\gamma\,{>}]\!] + \tilde{\mu}(z).!z.Y \ \mid\ Rec\ Y.\ \overline{\alpha} + \mu(\beta).!\beta.Y \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \rightarrow^{\pi}\ (by\ \mu)$

$Rec\ Y.\ \overline{x} + \tilde{\mu}(z).!z.Y \mid [\![y\cdot\gamma]\!] \mid env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \rightarrow^{\pi}\ (by\ x)$

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$Rec\ Y.\ \lambda(\delta'').[\![\tilde{\mu}z.{<}\, z \mid \delta'' \,{>}]\!] + \tilde{\mu}(z).\,!z.Y \ \mid\ Rec\ Y.\ \overline{\lambda}(\delta).([\![y]\!] \mid\ !\delta.[\![\gamma]\!]) + \mu(\beta).\,!\beta.Y \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \to^{\pi}\ (by\ \lambda)$
$Rec\ Y.\ \overline{y} + \tilde{\mu}(z).\,!z.Y \ \mid\ Rec\ Y.\ \overline{\tilde{\mu}}(z).[\![{<}\, z \mid \delta'' \,{>}]\!] + \mu(\beta).\,!\beta.Y \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \mid env(\delta'',[\![\gamma]\!]) \to^{\pi}\ (by\ \tilde{\mu})$
$Rec\ Y.\ \overline{z} + \tilde{\mu}(z).\,!z.Y \ \mid\ Rec\ Y.\ \overline{\delta''} + \mu(\beta).\,!\beta.Y \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \mid env(\delta'',[\![\gamma]\!]) \mid env(z,[\![y]\!])$
$\to^{\pi}\ (by\ z)$
$Rec\ Y.\ \overline{y} + \tilde{\mu}(y).\,!y.Y \ \mid\ Rec\ Y.\ \overline{\overline{\delta''}} + \mu(\beta).\,!\beta.Y \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \mid env(\delta'',[\![\gamma]\!]) \mid env(z,[\![y]\!])$
$\to^{\pi}\ (by\ y)$
$[\![M]\!] \ \mid\ Rec\ Y.\ \overline{\delta''} + \tilde{\mu}(\beta).\,!\beta.Y \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \mid env(\delta'',[\![\gamma]\!]) \mid env(z,[\![y]\!])$
$\to^{\pi}\ (by\ \delta'')$
$[\![M]\!] \mid [\![\gamma]\!] \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \mid env(\delta'',[\![\gamma]\!]) \mid env(z,[\![y]\!])$
$\to^{\pi}\ (by\ \gamma)$
$[\![M]\!] \mid [\![\alpha]\!] \mid$
$env(\beta,[\![M\cdot\alpha]\!]) \mid env(\delta,[\![\beta]\!]) \mid env(x,[\![\lambda x.x]\!]) \mid env(\delta',[\![\alpha]\!]) \mid env(y,[\![M]\!]) \mid env(\gamma,[\![\alpha]\!]) \mid env(\delta'',[\![\gamma]\!]) \mid env(z,[\![y]\!])$

Note that the final process is equivalent to $[\![M]\!] \mid [\![\alpha]\!]$ (by $\equiv^{\pi}$) since none of the other processes is accessible. The process $[\![c]\!]$ reducts thus to $[\![{<}\, M \mid \alpha \,{>}]\!]$, as expected since c reducts to $<M \mid \alpha>$. In the step $(*)$, and in the analogous steps, we could have chosen to perform a communication with channel μ and capture directly the variable δ'. The proof in Section 2 ensures that the result would not change.

2 Correctness of $[\![\cdot]\!]$

Our proof follows the same line of the proofs in [2] given by Milner. In what follows $P \nrightarrow^{\pi}$ means that there is no P' such that $P \to^{\pi} P'$. We write $\to^{\pi+}$ for the transitive closure of $\to^{\pi}$, $\to^{\pi*}$ for the transitive and reflexive closure of $\to^{\pi}$ and $P \downarrow^{\pi} P'$ means $P \to^{\pi*} P'$ and $P' \nrightarrow^{\pi}$. We use the notation $c \nrightarrow$ and $c \downarrow c'$ for commands in the analogous way.

Definition 2.1 (Relation R)**.** *Let R be a relation containing pairs (c, P), with c a command and P a process. Let $\tilde{v}$ be a sequence of $\bar{\lambda}\mu\tilde{\mu}$-terms. Let $\tilde{x}$ be a sequence of term variables (of $\bar{\lambda}\mu\tilde{\mu}$). Let $\tilde{e}$ be a sequence of contexts. Let $\tilde{\alpha}$ be a sequence of context variables.*
Let R contain all pairs such that:

$$c \equiv c'[\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$$

$$P \equiv^{\pi} [\![c']\!] \ \mid\ env(\tilde{x},\tilde{v}) \ \mid\ env(\tilde{\alpha},\tilde{e})$$

Assume variables are progressively numbered, i.e. $\tilde{x} = x_1, x_2, x_3, \ldots x_k$, the length of $\tilde{x}$ and $\tilde{v}$ sequences is k, the length of $\tilde{\alpha}$ and $\tilde{e}$ sequences is m. We write $env(x,v)$ for the process $!x.[\![v]\!]$ and we write $env(\tilde{x},\tilde{v})$ for $env(x_1,v_1) \mid env(x_2,v_2) \mid \ldots \mid env(x_k,v_k)$.

Moreover assume $FV(c') \subset \tilde{x} \cup \tilde{\alpha}$ and what follows:

- $FV(v_i) \subset \{x_{i+1}, x_{i+2}, x_{i+3}, \dots x_k\}$
- $FV(e_i) \subset \{e_{i+1}, e_{i+2}, e_{i+3}, \dots e_m\}$

Theorem 2.2 (correctness of the encoding)**.** *Let* $(c, P) \in R$:

- *(a) if* $c \nrightarrow$ *then* $P \downarrow^{\pi} P'$ *with* $(c, P') \in R$.
- *(b)* $c \to c'$ *then* $P \to^{\pi+} P'$ *with* $(c', P') \in R$.

Proof. Let $(c, P) \in R$. So $c = c'[\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$. Let y and β be variables not appearing in $\tilde{x}$ nor in $\tilde{\alpha}$, i.e. free in c. The proof proceeds by case analysis on c' and cope both (a) and (b) since either $c \nrightarrow$ or $c \to d$ for some d. For sake of readability we omit the initial sequence of restrictions in writing π-processes.

We firstly focus on normal forms.

- $c' =< y \mid \beta >$ (and $c =< y \mid \beta > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  $c \nrightarrow$
  $[\![c]\!] = Rec\ Y.\ \overline{y} + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\beta} + \mu(\alpha).!\alpha.Y \mid env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
  $[\![c]\!] \nrightarrow^{\pi}$ and $(c, [\![c]\!]) \in R$ by definition of R.
- $c' =< y \mid v_2 \cdot e >$ (and $c =< y \mid v_2 \cdot e > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  $c \nrightarrow$
  $[\![c]\!] = Rec\ Y.\ \overline{y} + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\lambda}(\delta).([\![v_2]\!] \mid !\delta.[\![e]\!]) + \mu(\alpha).!\alpha.Y \mid$
  $env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
  $[\![c]\!] \nrightarrow^{\pi}$ and $(c, [\![c]\!]) \in R$ by definition of R.
- $c' =< \lambda x.v \mid \beta >$ (and $c =< \lambda x.v \mid \beta > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  $c \nrightarrow$
  $[\![c]\!] = Rec\ Y.\ \lambda(\delta).[\![\tilde{\mu}x.< v_1 \mid \delta >]\!] + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\beta} + \mu(\alpha).!\alpha.Y \mid$
  $env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
  $[\![c]\!] \nrightarrow^{\pi}$ and $(c, [\![c]\!]) \in R$ by definition of R.

Next we focus on the cases for which a reduction takes place.

- $c' =< \lambda x.v_1 \mid v_2 \cdot e >$ (and $c =< \lambda x.v_1 \mid v_2 \cdot e > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  $c \to c'' =< v_2 \mid \tilde{\mu}x.< v_1 \mid e > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$
  $P = Rec\ Y.\ \lambda(\delta).[\![\tilde{\mu}x.< v_1 \mid \delta >]\!] + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\lambda}(\delta).([\![v_2]\!] \mid !\delta.[\![e]\!]) + \mu(\beta).!\beta.Y \mid$
  $env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
  $P \to^{\pi} P' = [\![v_2]\!] \mid [\![\tilde{\mu}x.< v_1 \mid \delta >]\!] \mid$
  $env(\delta, [\![e]\!]) \mid env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$

  $(c'', P') \in R$.
  Indeed $c'' \equiv< v_2 \mid \tilde{\mu}x.< v_1 \mid \delta > [e/\delta][\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$.
- $c' =< \mu\beta.c \mid e >$ (and $c =< \mu\beta.c \mid e > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  (*) Assume e is not a $\tilde{\mu}$-abstraction nor a variable appearing in $\tilde{\alpha}$, we treat those cases apart.
  $c \to c'' = c_1[e/\beta][\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$
  $P = Rec\ Y.\overline{\mu}(\beta).[\![c]\!] + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ Q\ +\ \mu(\beta).!\beta.Y \mid$
  $env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
  Q is the first operand of the main sum in $[\![e]\!]$. Given (*), no matter what the form e is it does not perform an output along the channel $\tilde{\mu}$ nor it can activate any process. The only possible reduction

is the following:
$P \rightarrow^{\pi} P' = [\![c]\!] \mid env(\beta, [\![e]\!]) \mid env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
$(c'', P') \in R$ by definition of R.

- $c' =< v \mid \tilde{\mu}x.c >$ (and $c =< v \mid \tilde{\mu}x.c > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
(*) Assume v is not a μ-abstraction nor a variable in $\tilde{x}$, we treat those cases apart. The proof is analogous to the previous one.

- $c' =< \mu\beta.c_1 \mid \tilde{\mu}x.c_2 >$ (and $c =< \mu\beta.c_1 \mid \tilde{\mu}x.c_2 > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
$P = Rec\ Y.\ \overline{\mu}(\beta).[\![c]\!] + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\tilde{\mu}}(x).[\![c_1]\!] + \mu(\beta).!\beta.Y \mid$
$env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
Two possible reductions: by cases
(1)
$c \rightarrow c'' = c_1[\tilde{\mu}x.c_2/\beta][\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$
$P \rightarrow^{\pi} P' = [\![c_1]\!] \mid env(\tilde{\mu}x.c_2, \beta) \mid env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
$(c'', P') \in R$ by definition of R.

(2)
$c \rightarrow c'' = c_2[\mu\beta.c_1/x][\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$
$P \rightarrow^{\pi} P' = [\![c_2]\!] \mid env(\mu\beta.c_1, x) \mid env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$
$(c'', P') \in R$ by definition of R.

Next we focus on the cases where a bound variable is involved and an access to an environment entry may occur.

- $c' =< x_i \mid e >$ (and $c =< x_i \mid e > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
(*) Assume e is not a $\tilde{\mu}$-abstraction nor a variable in $\tilde{\alpha}$, we treat those cases apart.
Proof by complete induction on $k - i$
$c =< x_i \mid e > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$
$P = Rec\ Y.\ \overline{x}_i + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ Q + \tilde{\mu}(\beta).!\beta.Y \mid$
$env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$ (recall that $env(x_i, v_i) = !x_i.[\![v_i]\!]$)
Q is the first operand of the main sum in $[\![e]\!]$. Given (*), no matter what the form e is it does not perform an output along the channel $\tilde{\mu}$ nor it can activate any process. The only possible reduction is the following:
$P \rightarrow^{\pi} P' = [\![< v_i \mid e >]\!] \mid env(\tilde{x}, \tilde{v}) \mid env(\tilde{\alpha}, \tilde{e})$

If v_i is a variable appearing in $\tilde{x}$, say x_j, then $j > i$ and $k - j < k - i$, thus we apply the induction hypothesis. If v_i is not a variable appearing in $\tilde{x}$ then its form matches another proved case of the proof.

- $c' =< v \mid \alpha_j >$ (and $c =< v \mid \alpha_j > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
Assume v is not a μ-abstraction nor a variable in $\tilde{x}$, we treat those cases apart. The proof is analogous to the previous one (the induction is on $m - i$).

- $c' =< x_i \mid \alpha_j >$ (and $c =< x_i \mid \alpha_j > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
The proof is a straightforward combination of the schema of the previous two cases.

- $c' =< \mu\beta.\, c_1 \mid \alpha_i >$ (and $c =< \mu\beta.\, c_1 \mid \alpha_i > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  $P = Rec\ Y.\ \overline{\mu}(\beta).[\![c_1]\!] + \tilde{\mu}(x).!x.Y \mid Rec\ Y.\ \overline{\alpha}_i + \mu(\beta).!\beta.Y \mid env(\tilde{x},\tilde{v}) \mid env(\tilde{\alpha},\tilde{e})$
  Two possible reductions: by cases
  (1)
  $c \to c'' = c_1[\alpha_i/\beta][\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$
  $P \to^{\pi} P' = [\![c_1]\!] \mid env(\alpha_i,\beta) \mid env(\tilde{x},\tilde{v}) \mid env(\tilde{\alpha},\tilde{e})$
  $(c'', P') \in R$ by definition of R.

  (2)
  $P \to^{\pi} P' = [\![\mu\beta.\, c_1]\!] \mid [\![e_i]\!] \mid env(\tilde{x},\tilde{v}) \mid env(\tilde{\alpha},\tilde{e})$
  Proof by complete induction on $m - i$

  If e_i is a variable appearing in $\tilde{\alpha}$, say α_j, then $j > i$ and $m - j < m - i$, thus we apply the induction hypothesis. If e_i is not a variable appearing in $\tilde{\alpha}$ then its form matches another proved case of the proof.
- $c' =< x_i \mid \tilde{\mu}y.\, c >$ (and $c =< x_i \mid \tilde{\mu}y.\, c > [\tilde{v}/\tilde{x},\ \tilde{e}/\tilde{\alpha}]$)
  The proof is analogous to the previous one (the induction is on $k - i$).

□

The encoding intimately mimicks $\bar{\lambda}\mu\tilde{\mu}$. Indeed as a consequence of Theorem 2.2 either one of the following conditions holds:

- $c \uparrow$ and $[\![c]\!] \uparrow$
- $c \downarrow c_1$ and $[\![c]\!] \downarrow P$ with $(c_1, P) \in R$

3 $[\![\cdot]\!]$ is an operational full abstraction (correspondence π-$M1$)

Let ρ be an environment that binds the variables $\tilde{y}$. Assume that x and β are not appearing in $\tilde{y}$. In what follows $s \not\longrightarrow^{M1}$ means that there is no s' such that $s \longrightarrow^{M1} s'$.

Proof. $\Rightarrow$
The proof proceeds by case analysis on the states of $M1$.
Normal forms:

- $< x \mid \beta >\ in\ \rho \not\longrightarrow^{M1}$
  $P = [\![< x \mid \beta >\ in\ \rho]\!] = \nu\tilde{y}.(Rec\ Y.\ \overline{x} + \tilde{\mu}(z).!z.Y \mid\ Rec\ Y.\ \overline{\beta} + \mu(\alpha).!\alpha.Y \mid [\![\rho]\!])$
  $P \not\twoheadrightarrow^{\pi}$
- $< x \mid v_2 \cdot e >\ in\ \rho \not\longrightarrow^{M1}$
  $P = [\![< x \mid v_2 \cdot e >\ in\ \rho]\!] = \nu\tilde{y}.(Rec\ Y.\ \overline{x} + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\lambda}(\delta).([\![v_2]\!] \mid !\delta.[\![e]\!]) + \mu(\beta).!\beta.Y \mid [\![\rho]\!])$
  $P \not\twoheadrightarrow^{\pi}$
- $< \lambda x.\, v \mid \beta >\ in\ \rho \not\longrightarrow^{M1}$
  $P = [\![< \lambda x.\, v \mid \beta >\ in\ \rho]\!] = \nu\tilde{y}.(Rec\ Y.\ \lambda(\delta).[\![\tilde{\mu}x.< v_1 \mid \delta >]\!] + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\beta} + \mu(\alpha).!\alpha.Y \mid [\![\rho]\!])$
  $P \not\twoheadrightarrow^{\pi}$

Reduction cases:

- $< \mu\beta.c \mid e > \ in\ \rho \longrightarrow^{M1} c[\beta'/\beta]\ in\ \rho\ +\ [\beta' = e]$
  $P = [\![< \mu\beta.c \mid e > \ in\ \rho]\!] = \nu\tilde{y}.(Rec\ Y.\ \overline{\mu}(\beta).[\![c]\!] + \tilde{\mu}(x).!x.Y \mid Rec\ Y.Q + \mu(\beta).!\beta.Y \mid [\![\rho]\!])$
  $P \rightarrow^{\pi} P' \equiv^{\pi} \nu\beta'.\nu\tilde{y}.([\![c]\!][\beta'/\beta] \mid !\beta'.[\![e]\!] \mid [\![\rho]\!])$ and $P' \equiv^{\pi} [\![c[\beta'/\beta]\ in\ \rho\ +\ [\beta' = e]]\!]$
  (a simple α-conversion is sufficient)
- $< v \mid \tilde{\mu}x.c > \ in\ \rho \longrightarrow^{M1} c[x'/x]\ in\ \rho\ +\ [x' = v]$
  The proof is analogous to the previous one.
- $< x \mid e > \ in\ \rho\ +\ [x = v] \longrightarrow^{M1} < v \mid e > \ in\ \rho\ +\ [x = v]$
  $P = [\![< x \mid e > \ in\ \rho\ +\ [x = v]]\!] = \nu x.\nu\tilde{y}.(Rec\ Y.\ \overline{x} + \tilde{\mu}(z).!z.Y \mid [\![e]\!] \mid !x.[\![v]\!] \mid [\![\rho]\!])$
  $P \rightarrow^{\pi} P' \equiv^{\pi} \nu x.\nu\tilde{y}.([\![v]\!] \mid [\![e]\!] \mid !x.[\![v]\!] \mid [\![\rho]\!])$ and $P' \equiv^{\pi} [\![< v \mid e > \ in\ \rho\ +\ [x = v]]\!]$
- $< v \mid \beta > \ in\ \rho\ +\ [\beta = e] \longrightarrow^{M1} < v \mid e > \ in\ \rho\ +\ [\beta = e]$
  The proof is analogous to the previous one.
- $< \lambda x.v_1 \mid (v_2 \cdot e) > \ in\ \rho \longrightarrow^{M1} < v_2 \mid \tilde{\mu}x.< v_1 \mid \delta >> \ in\ \rho\ +\ [\delta = e] >>$
  $P = [\![< \lambda x.v_1 \mid (v_2 \cdot e) > \ in\ \rho]\!] = \nu\tilde{y}.(Rec\ Y.\ \lambda(\delta).[\![\tilde{\mu}x.< v_1 \mid \delta >]\!] + \tilde{\mu}(z).!z.Y \mid$
  $Rec\ Y.\ \overline{\lambda}(\delta).([\![v_2]\!] \mid !\delta.[\![e]\!]) + \mu(\beta).!\beta.Y \mid [\![\rho]\!])$
  $P \rightarrow^{\pi} P' \equiv^{\pi} \nu\delta.\nu\tilde{y}.([\![v_2]\!] \mid [\![\tilde{\mu}x.< v_1 \mid \delta >]\!] \mid !\delta.[\![e]\!] \mid [\![\rho]\!])$
  and $P' \equiv^{\pi} [\![< v_2 \mid \tilde{\mu}x.< v_1 \mid \delta >> \ in\ \rho\ +\ [\delta = e] >>]\!]$

□

Proof. $\Leftarrow$

Since we reason by cases, the proof for $\Rightarrow$ takes into account the same cases for $\Leftarrow$, thus the deterministic cases are already proved (it is easy to check that the encodings are determinstic too). The same consideration holds for normal forms. What we need to check is whether the remaining transitions lead to what we expect in $M1$ for every possible π step.

The non deterministic cases are $[\![< \mu\beta.c \mid \alpha > \ in\ \rho\ +\ [\alpha = e]]\!]$, $[\![< x \mid \tilde{\mu}y.c > \ in\ \rho\ +\ [x = v]]\!]$ and $[\![< \mu\beta.c_1 \mid \tilde{\mu}x.c_2 > \ in\ \rho]\!]$.

- $s =< \mu\beta.c \mid \alpha > \ in\ \rho\ +\ [\alpha = e]$
  $P = [\![s]\!] = \nu\alpha.\nu\tilde{y}.(Rec\ Y.\ \overline{\mu}(\beta).[\![c]\!] + \tilde{\mu}(x).!x.Y \mid Rec\ Y.\overline{\alpha} + \mu(\beta).!\beta.Y \mid !\alpha.[\![e]\!] \mid [\![\rho]\!]$

  Two reduction cases:
  (1) $P \rightarrow^{\pi} P' \equiv^{\pi} \nu\beta'.\nu\alpha.\nu\tilde{y}.([\![c]\!][\beta'/\beta] \mid !\beta'.[\![\alpha]\!] \mid !\alpha.[\![e]\!] \mid [\![\rho]\!])$
  (a simple α-conversion)
  $P' \equiv^{\pi} [\![c[\beta'/\beta]\ in\ [\beta' = \alpha] + [\alpha = e] + \rho]\!]$
  indeed $s \longrightarrow^{M1} \equiv^{M1} c[\beta'/\beta]\ in\ [\beta' = \alpha] + [\alpha = e] + \rho$
  (2) $P \rightarrow^{\pi} P' = \nu\alpha.\nu\tilde{y}.(Rec\ Y.\ \overline{\mu}(\beta).[\![c]\!] + \tilde{\mu}(x).!x.Y \mid [\![e]\!] \mid [\![\rho]\!]$
  $P' \equiv^{\pi} [\![< \mu\beta.c \mid e > \ in\ [\alpha = e] + \rho]\!]$
  indeed $s \longrightarrow^{M1} \equiv^{M1} < \mu\beta.c \mid e > \ in\ [\alpha = e] + \rho$
- $s =< x \mid \tilde{\mu}y.c > \ in\ \rho\ +\ [x = v]$
  The proof is analogous to the previous one.
- $s =< \mu\beta.c_1 \mid \tilde{\mu}x.c_2 > \ in\ \rho$
  $P = [\![s]\!] = \nu\tilde{y}.(Rec\ Y.\ \overline{\mu}(\beta).[\![c]\!] + \tilde{\mu}(z).!z.Y \mid Rec\ Y.\ \overline{\mu}(x).[\![c]\!] + \mu(\alpha).!\alpha.Y \mid [\![\rho]\!])$

Two reduction cases:

(1) $P \rightarrow^{\pi} P' \equiv^{\pi} \nu\beta' . \nu\tilde{y} . ([\![c_1]\!] \mid\ !\beta' . [\![\tilde{\mu}x . c_2]\!] \mid\ [\![\rho]\!])$
(a simple α-conversion)
$P' \equiv^{\pi} [\![c_1[\beta'/\beta]\ in\ [\beta' = \tilde{\mu}x . c_2] + \rho]\!]$
indeed $s \longrightarrow^{M1} \equiv^{M1} c_1[\beta'/\beta]\ in\ [\beta' = \tilde{\mu}x . c_2] + \rho$

(2) $P \rightarrow^{\pi} P' \equiv^{\pi} \nu x' . \nu\tilde{y} . ([\![c_2]\!] \mid\ !x' . [\![\mu\beta . c_1]\!] \mid\ [\![\rho]\!])$
(a simple α-conversion)
$P' \equiv^{\pi} [\![c_2[x'/x]\ in\ [x' = \mu\beta . c_1] + \rho]\!]$
indeed $s \longrightarrow^{M1} \equiv^{M1} c_2[x'/x]\ in\ [x' = \mu\beta . c_1] + \rho$

□

4 $[\![\cdot]\!]^M$ is an operational full abstraction (correspondence $M1$-$M2$)

Before embarking in the proof, we define explicitly the mapping $[\![\cdot]\!]^M$ from states of $M2$ to states of $M1$. Intuitively, $[\![\cdot]\!]^M$ maps the explicit substitutions of $M2$ into the global environment of $M1$, caring that name clashes are avoided.

$$\begin{array}{rcl} [\![< v\ in\ \rho_1 \mid e\ in\ \rho_2 >]\!]^M & = & < v \mid e >\ in\ [\![\rho_1]\!]^M\ +\ [\![\rho_2]\!]^M \\ [\![\rho]\!]^M & = & \sum [var_i = t_i]\ +\ \sum [\![\phi_i]\!]^M \\ with\ \rho & = & [var_1 = t_1\ in\ \phi_1, var_2 = t_2\ in\ \phi_2 \ldots\ var_n = t_n\ in\ \phi_n] \end{array}$$

In what follows $s \not\longrightarrow^{M1}$ means that there is no s' such that $s \longrightarrow^{M1} s'$. We write $q \not\longrightarrow^{M2}$ analogously.

Proof. $\Rightarrow$
The proof proceeds by case analysis on the states of $M2$. Assume the variables x and β not bound by their environments when they occur.

Normal forms:

- $< x\ in\ \rho_1 \mid \beta\ in\ \rho_2 >\ \not\longrightarrow^{M2}$
  $s = [\![< x\ in\ \rho_1 \mid \beta\ in\ \rho_2 >]\!]^M =< x \mid \beta >\ in\ [\![\rho_1]\!]^M\ +\ [\![\rho_2]\!]^M \not\longrightarrow^{M1}$

- $< x\ in\ \rho_1 \mid (v_2 \cdot e)\ in\ \rho_2 >\ \not\longrightarrow^{M2}$
  $s = [\![< x\ in\ \rho_1 \mid (v \cdot e)\ in\ \rho_2 >]\!]^M =< x \mid (v \cdot e) >\ in\ [\![\rho_1]\!]^M\ +\ [\![\rho_2]\!]^M \not\longrightarrow^{M1}$

- $< \lambda x . v\ in\ \rho_1 \mid \beta\ in\ \rho_2 >\ \not\longrightarrow^{M2}$
  $s = [\![< \lambda x . v\ in\ \rho_1 \mid \beta\ in\ \rho_2 >]\!]^M =< \lambda x . v \mid \beta >\ in\ [\![\rho_1]\!]^M\ +\ [\![\rho_2]\!]^M \not\longrightarrow^{M1}$

Reduction cases:

- $< \mu\beta . c\ in\ \rho_1 \mid e\ in\ \rho_2 >\ \longrightarrow^{M2} c\ in\ \rho_1 + [\beta = e\ in\ \rho_2] \equiv^{M2} c[\beta'/\beta]\ in\ \rho_1 + [\beta' = e\ in\ \rho_2]$
  $[\![< \mu\beta . c\ in\ \rho_1 \mid e\ in\ \rho_2 >]\!]^M =< \mu\beta . c \mid e >\ in\ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M >\ \longrightarrow^{M1}$
  $c[\beta'/\beta]\ in\ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M + [\beta' = e] \equiv^{M1} [\![c[\beta'/\beta]\ in\ \rho_1 + [\beta' = e\ in\ \rho_2]]\!]^M$

- $< v \ in \ \rho_1 \ | \ \tilde{\mu}x.c \ in \ \rho_2 > \longrightarrow^{M2} c \ in \ \rho_2 + [x = v \ in \ \rho_1]$
  The proof is analogous to the previous one.
- $< x \ in \ \rho_1 + [x = v \ in \ \rho_2] \ | \ e \ in \ \rho_3 > \ \longrightarrow^{M2} \ < v \ in \ \rho_2 \ | \ e \ in \ \rho_3 >$
  $[\![< x \ in \ \rho_1 + [x = v \ in \ \rho_2] \ | \ e \ in \ \rho_3 >]\!]^M =< x \ | \ e > \ in \ [\![\rho_1]\!]^M + [\![\rho_3]\!]^M + [x = v] + [\![\rho_2]\!]^M > \longrightarrow^{M1}$
  $< v \ | \ e > \ in \ [\![\rho_1]\!]^M + [\![\rho_3]\!]^M + [x = v] + [\![\rho_2]\!]^M \equiv^{M1} [\![< v \ in \ \rho_2 \ | \ e \ in \ \rho_3 >]\!]^M$
  Indeed ρ_1 and $[x = v]$ are no longer accessible.
- $< v \ in \ \rho_1 \ | \ \beta \ in \ \rho_2 + [\beta = e \ in \ \rho_3] > \longrightarrow^{M2} < v \ in \ \rho_1 \ | \ e \ in \ \rho_3 >$
  The proof is analogous to the previous one.
- $< \lambda x.v_1 \ in \ \rho_1 \ | \ v_2 \cdot e \ in \ \rho_2 > \ \longrightarrow^{M2} \ < v_2 \ in \ \rho_2 \ | \ \tilde{\mu}x. < v_1 \ in \ \rho_1 \ | \ \delta \ in \ [\delta = e \ in \ \rho_2] >>$
  $[\![< \lambda x.v_1 \ in \ \rho_1 \ | \ v_2 \cdot e \ in \ \rho_2 >]\!]^M =< \lambda x.v_1 \ | \ v_2 \cdot e > \ in \ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M > \longrightarrow^{M1}$
  $< v_2 \ | \ \tilde{\mu}x. < v_1 \ | \ \delta >> \ in \ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M + [\delta = e] \equiv^{M1} [\![< v_2 \ in \ \rho_2 \ | \ \tilde{\mu}x. < v_1 \ in \ \rho_1 \ | \ \delta \ in \ [\delta = e \ in \ \rho_2] >>]\!]^M$

□

Proof. ⇐

Since we reason by cases, the proof for ⇒ takes into account the same cases for ⇐, thus the deterministic cases are already proved (it is easy to check that the encodings are determinstic too). The same consideration holds for normal forms. What we need to check is whether the remaining transitions lead to what we expect in $M2$ for every possible $M1$ step.

The non deterministic cases are $[\![< \mu\beta.c \ in \ \rho_1 \ | \ \alpha \ in \ [\alpha = e \ in \ \rho_2] \ + \ \rho_3 >]\!]^M$, $[\![< x \ in \ [x = v \ in \ \rho_1] + \rho_2 \ | \ \mu y.c \ in \ \rho_3 >]\!]^M$, $[\![< \mu\beta.c_1 \ in \ \rho_1 \ | \ \mu y.c_2 \ in \ \rho_2 >>]\!]^M$.

- $s =< \mu\beta.c \ in \ \rho_1 \ | \ \alpha \ in \ [\alpha = e \ in \ \rho_2] \ + \ \rho_3 >$
  $q = [\![s]\!]^M =< \mu\beta.c \ | \ \alpha > \ in \ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M + [\alpha = e] + [\![\rho_3]\!]^M >$
  Two reduction cases:
  (1) $q \longrightarrow^{M1} q' = c[\beta/\beta] \ in \ [\beta' = \alpha] + [\![\rho_1]\!]^M + [\![\rho_2]\!]^M + [\alpha = e] + [\![\rho_3]\!]^M >$
  $q' = [\![c[\beta'/\beta] \ in \ [\beta' = \alpha \ in \ [\alpha = e \ in \ \rho_2] \ + \ \rho_3]]\!]^M$
  indeed $s \longrightarrow^{M2} \equiv^{M2} c[\beta'/\beta] \ in \ [\beta' = \alpha \ in \ [\alpha = e \ in \ \rho_2] \ + \ \rho_3]$
  (2) $q \longrightarrow^{M1} q' =< \mu\beta.c \ | \ e > \ in \ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M + [\alpha = e] + [\![\rho_3]\!]^M >$
  $q' \equiv^{M1} [\![< \mu\beta.c \ in \ \rho_1 \ | \ e \ in \ \rho_2 >]\!]$
  indeed $s \longrightarrow^{M2} \equiv^{M2} < \mu\beta.c \ in \ \rho_1 \ | \ e \ in \ \rho_2 >$
- $s =< x \ in \ [x = v \ in \ \rho_1] + \rho_2 \ | \ \mu y.c \ in \ \rho_3 >$
  The proof is analogous to the previous one.
- $s =< \mu\beta.c_1 \ in \ \rho_1 \ | \ \mu y.c_2 \ in \ \rho_2 >>$
  $q = [\![s]\!]^M =< \mu\beta.c \ | \ \mu y.c > \ in \ [\![\rho_1]\!]^M + [\![\rho_2]\!]^M >$
  Two reduction cases:
  (1) $q \longrightarrow^{M1} q' = c_1[\beta'/\beta] \ in \ [\beta' = \mu y.c_2] + [\![\rho_1]\!]^M + [\![\rho_2]\!]^M >$
  $q' = [\![c_1[\beta'/\beta] \ in \ [\beta' = \mu y.c_2 \ in \ \rho_2] + \rho_1]\!]^M$
  indeed $s \longrightarrow^{M2} \equiv^{M2} c_1[\beta'/\beta] \ in \ [\beta' = \mu y.c_2 \ in \ \rho_2] + \rho_1$

  (2) $q \longrightarrow^{M1} q' = c_2[x'/x] \ in \ [x' = \mu\beta.c_1] + [\![\rho_1]\!]^M + [\![\rho_2]\!]^M$
  $q' = [\![c_2[x'/x] \ in \ [x' = \mu\beta.c_1] \ in \ \rho_1] + \rho_2]\!]^M$

indeed $s \longrightarrow^{M2} \equiv^{M2} c_2[x'/x]\ in\ [x' = \mu\beta.c_1]\ in\ \rho_1] + \rho_2$

□

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